

FORMAL GAGA FOR GOOD MODULI SPACES

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ABSTRACT. We prove formal GAGA for good moduli space morphisms under an assumption of “enough vector bundles” (which holds for instance for quotient stacks). This supports the philosophy that though they are non-separated, good moduli space morphisms largely behave like proper morphisms.

Good moduli space morphisms generalize the notion of good quotients [GIT] and are moreover a common generalization of tame Artin stacks [AOV08, Definition 3.1] and coarse moduli spaces [FC90, Theorem 4.10].

Definition ([Alp09, Definition 4.1]). A quasi-compact and quasi-separated morphism of locally Noetherian algebraic stacks $\phi: \mathcal{X} \rightarrow \mathcal{Y}$ is a *good moduli space morphism* if

- (ϕ is *Stein*) the morphism $\mathcal{O}_{\mathcal{Y}} \rightarrow \phi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism, and
- (ϕ is *cohomologically affine*) the functor $\phi_*: \mathrm{QCoh}(\mathcal{O}_{\mathcal{X}}) \rightarrow \mathrm{QCoh}(\mathcal{O}_{\mathcal{Y}})$ is exact.

Any morphism from \mathcal{X} to an algebraic space factors uniquely through such a ϕ .¹ In particular, if there exists a good moduli space morphism $\phi: \mathcal{X} \rightarrow X$ where X is an algebraic space, then X is determined up to unique isomorphism. In this case, X is said to be *the good moduli space* of \mathcal{X} . If $\mathcal{X} = [U/G]$, this corresponds to X being a good quotient of U by G in the sense of [GIT] (e.g. for a linearly reductive G , $[\mathrm{Spec} R/G] \rightarrow \mathrm{Spec} R^G$ is a good moduli space).

In many respects, good moduli space morphisms behave like proper morphisms. They are universally closed [Alp09, Theorem 4.16(ii)] and weakly separated [ASvdW10, Proposition 2.17], but since points of \mathcal{X} can have non-proper stabilizer groups, good moduli space morphisms are generally not separated (e.g. if G is a non-proper group scheme, BG is not separated). Pushforward along a good moduli space morphism respects coherence [Alp09, Theorem 4.16(x)]. The main theorem in this paper continues this philosophy, showing that formal GAGA holds for good moduli space morphisms, at least when the stack has “enough vector bundles.” Recall that a stack is said to have *the resolution property* if every coherent sheaf has a surjection from a vector bundle.

Theorem 1. *Suppose $\mathcal{X} \rightarrow \mathrm{Spec} A$ is a good moduli space, where A is a complete Noetherian local ring with maximal ideal \mathfrak{m} and \mathcal{X} is finite type over $\mathrm{Spec} A$. Let $\hat{\mathcal{X}}$ denote the formal completion of \mathcal{X} with respect to \mathfrak{m} (see §1).*

- The completion functor $\mathrm{Coh}(\mathcal{X}) \rightarrow \mathrm{Coh}(\hat{\mathcal{X}})$ is fully faithful.*
- Suppose $\mathcal{X}_0 = \mathcal{X} \times_{\mathrm{Spec} A} \mathrm{Spec} A/\mathfrak{m}$ has the resolution property (e.g. \mathcal{X}_0 is a quotient stack; see Remark 15). Then the following conditions are equivalent:*
 - (GAGA) *The completion functor $\mathrm{Coh}(\mathcal{X}) \rightarrow \mathrm{Coh}(\hat{\mathcal{X}})$ is an equivalence.*
 - (res) *\mathcal{X} has the resolution property.*
 - (res') *Every coherent sheaf on \mathcal{X}_0 has a surjection from a vector bundle on \mathcal{X} .*

The above conditions are implied by the equivalent conditions below. If the unique closed point²

¹Since algebraic spaces are sheaves in the smooth topology, this property may be checked smooth locally on \mathcal{Y} . Good moduli space morphisms are stable under base change [Alp09, Proposition 4.7(i)], so we may assume \mathcal{Y} is an algebraic space, in which case the result is [Alp09, Theorem 6.6].

²By [Alp09, Theorem 4.16(iii)] and Proposition 9.1], if $\mathcal{X} \rightarrow X$ is a good moduli space morphism and X has a unique closed point, then \mathcal{X} also has a unique closed point.

of \mathcal{X} has affine stabilizer group, all five conditions are equivalent.

(quot) \mathcal{X} is the quotient of an affine scheme by GL_n for some n .

(quot') \mathcal{X} is the quotient of an algebraic space by an affine algebraic group.

In Conjecture 28, we predict that if \mathcal{X} has affine diagonal, then (GAGA) holds (though \mathcal{X}_0 may not have the resolution property). We provide examples in §4 to show that (GAGA) may fail under weaker hypotheses.

Remark 2. In [Ols05, Theorem 1.4] (see also [Con]), Olsson proves that formal GAGA holds for proper Artin stacks. His main theorem gives a proper surjection from a proper scheme $X \rightarrow \mathcal{X}$, and formal GAGA follows from a dévissage (as outlined in [HR10, §1.2]). In our setting such a surjection does not exist, and our arguments are quite different.

Remark 3. If \mathcal{X} has quasi-compact and quasi-separated diagonal over a base S , the Hilbert stack $\underline{HS}_{\mathcal{X}/S}$ of quasi-finite maps from proper schemes is an algebraic stack [HR10, Theorem 2]. A key ingredient in the proof of this result is a weaker variant of formal GAGA for non-separated stacks: though the completion functor fails to be an equivalence of categories of coherent sheaves, it induces an equivalence between the larger (non-abelian) categories of pseudosheaves.

Remark 4. Formal GAGA allows the study of a stack \mathcal{X} with good moduli space X to largely be reduced to the study of the fibers of the map $\mathcal{X} \rightarrow X$. This reduction is particularly appealing since it is possible that the geometric fibers of this map must be quotient stacks (see Question 31 and Remark 34). Here is the template for the reduction:

- (0) *Start with a problem which is étale local on X , and a solution to the problem for the fiber over a point x .*
- (1) *Use deformation theory to extend the solution to a formal solution.* Deformation theory typically shows that the problem of extending a solution from an infinitesimal neighborhood to a larger infinitesimal neighborhood is controlled by the cohomologies of certain quasi-coherent sheaves. An immediate consequence of $\mathcal{X} \rightarrow \mathrm{Spec} A$ being cohomologically affine is that all such higher cohomology groups vanish, so deformation-theoretic problems are extremely easy when working with good moduli space morphisms. (See Lemma 13 as an example of this.)
- (2) *Show that any formal solution is effectivizable.* That is, show that any compatible family of solutions over all infinitesimal neighborhoods of $x \in X$ is induced by a solution over $\mathrm{Spec} \hat{\mathcal{O}}_{X,x}$. If the question can be formulated entirely in terms of coherent sheaves, as is often the case, then (GAGA) does this step.
- (3) *Use Artin approximation [Art69, Theorem 1.12] to extend the solution to an étale neighborhood of x .* If the stack of solutions is locally finitely presented, Artin's theorem says that for a map f from the complete local ring at a point, there is a map from the henselization of the local ring which agrees with f modulo any given power of the maximal ideal. (By [LO09, Proposition 2.3.8], one can instead apply Artin's theorem to the associated functor of isomorphism classes.) By step 1 (uniqueness of deformations) and formal GAGA, this must actually be an extension of f . By local finite presentation, this map extends to some étale neighborhood, as the henselization is the limit of all étale neighborhoods.

Proposition 29 illustrates this template. It shows that if $\mathcal{X} \rightarrow X$ a good moduli space, $x \in X$ is a point at which formal GAGA holds, and the fiber over x is a quotient stack, then there is some étale neighborhood of x over which \mathcal{X} is a quotient stack.

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1. TERMINOLOGY

Throughout the paper \mathcal{X} is an algebraic stack with good moduli space $\mathcal{X} \rightarrow \operatorname{Spec} A$, where A is a complete Noetherian local ring with maximal ideal \mathfrak{m} , and the map $\mathcal{X} \rightarrow \operatorname{Spec} A$ is of finite type. We denote by $\mathcal{X}_{\text{lis-ét}}$ the lisse-étale topos of \mathcal{X} and define $\widehat{\mathcal{O}}_{\mathcal{X}}$ to be the completion $\varprojlim \mathcal{O}_{\mathcal{X}}/\mathcal{I}^n$, where \mathcal{I} is the sheaf of ideals generated by the pullback of $\mathfrak{m} \subseteq A$. Following [Con, §1], we define the ringed topos $\widehat{\mathcal{X}}$ to be the pair $(\mathcal{X}_{\text{lis-ét}}, \widehat{\mathcal{O}}_{\mathcal{X}})$. There is a natural completion functor

$$\operatorname{Coh}(\mathcal{X}) \rightarrow \operatorname{Coh}(\widehat{\mathcal{X}}), \quad \mathcal{F} \mapsto \widehat{\mathcal{F}} := \varprojlim \mathcal{F}/\mathcal{I}^{n+1}\mathcal{F}.$$

Letting $\mathcal{X}_n = \mathcal{X} \times_{\operatorname{Spec} A} \operatorname{Spec} A/\mathfrak{m}^{n+1}$, the natural functor $\operatorname{Coh}(\widehat{\mathcal{X}}) \rightarrow \varprojlim \operatorname{Coh}(\mathcal{X}_n)$ is an equivalence of categories [Con, Theorem 2.3], where the map $\operatorname{Coh}(\mathcal{X}_n) \rightarrow \operatorname{Coh}(\mathcal{X}_{n-1})$ is given by pullback along the closed immersion $\mathcal{X}_{n-1} \rightarrow \mathcal{X}_n$. We may therefore regard elements of $\operatorname{Coh}(\widehat{\mathcal{X}})$ as compatible systems of coherent sheaves on the \mathcal{X}_n .

2. PROOF OF THEOREM 1

This section is quite technically involved. Subsequent sections depend on the results but not on the techniques or terminology developed in this section.

We use the terminology of topoi developed in [SGA4]. A morphism of topoi $f: Y \rightarrow X$ is a triple (f_*, f^{-1}, α) , where $f^{-1}: X \rightarrow Y$ is a functor which commutes with finite limits, $f_*: Y \rightarrow X$ is a functor, and α is an adjunction $\operatorname{Hom}_Y(f^{-1}(-), -) \xrightarrow{\sim} \operatorname{Hom}_X(-, f_*(-))$. If \mathcal{O}_Y and \mathcal{O}_X are sheaves of rings on Y and X , respectively, then a morphism of ringed topoi (also denoted $f: Y \rightarrow X$) is a morphism of topoi, together with a morphism of sheaves of rings $f^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_Y$. In this case, $f_*: \mathcal{O}_Y\text{-mod} \rightarrow \mathcal{O}_X\text{-mod}$ has the left adjoint $f^*(-) = f^{-1}(-) \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y$.

Definition 5. A morphism of ringed topoi $f: Y \rightarrow X$ is *flat* if f^* is exact.

Lemma 6. *If $f: Y \rightarrow X$ is a flat morphism of ringed topoi, \mathcal{F} is a quasi-coherent \mathcal{O}_X -module, and \mathcal{G} is any \mathcal{O}_X -module, then the natural map $f^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{H}om_{\mathcal{O}_Y}(f^*\mathcal{F}, f^*\mathcal{G})$ is an isomorphism.*

Proof. Case 1: If $\mathcal{F} \cong \mathcal{O}_X$, the natural map is simply the identity map on $f^*\mathcal{G}$. Similarly, if $\mathcal{F} \cong \mathcal{O}_X^{\oplus I}$, the map is the identity map on $f^*\mathcal{G}^{\oplus I}$.

Case 2: Suppose \mathcal{F} has a global presentation

$$\mathcal{O}_X^{\oplus J} \rightarrow \mathcal{O}_X^{\oplus I} \rightarrow \mathcal{F} \rightarrow 0.$$

Since f^* is right exact, we get a global presentation

$$\mathcal{O}_Y^{\oplus J} \rightarrow \mathcal{O}_Y^{\oplus I} \rightarrow f^*\mathcal{F} \rightarrow 0.$$

Applying $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{G})$ to the first sequence and $\mathcal{H}om_{\mathcal{O}_Y}(-, f^*\mathcal{G})$ to the second, we get the exact sequences

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus I}, \mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus J}, \mathcal{G})$$

$$0 \rightarrow \mathcal{H}om_{\mathcal{O}_Y}(f^*\mathcal{F}, f^*\mathcal{G}) \rightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y^{\oplus I}, f^*\mathcal{G}) \rightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y^{\oplus J}, f^*\mathcal{G}).$$

Since f is flat, the first sequence remains exact if we apply f^* , so the rows in the following diagram are exact. The squares commute by naturality of the vertical arrows.

$$\begin{array}{ccccc} 0 \rightarrow f^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) & \rightarrow & f^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus I}, \mathcal{G}) & \rightarrow & f^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus J}, \mathcal{G}) \\ \downarrow & & \downarrow \wr & & \downarrow \wr \\ 0 \rightarrow \mathcal{H}om_{\mathcal{O}_Y}(f^*\mathcal{F}, f^*\mathcal{G}) & \rightarrow & \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y^{\oplus I}, f^*\mathcal{G}) & \rightarrow & \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y^{\oplus J}, f^*\mathcal{G}). \end{array}$$

We've already shown that the middle and right vertical arrows are isomorphisms, so the left vertical arrow must also be an isomorphism, completing the proof in the case where \mathcal{F} is globally presented.

Case 3: Now we prove the general case. To check that the natural map $f^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{H}om_{\mathcal{O}_Y}(f^*\mathcal{F}, f^*\mathcal{G})$ is an isomorphism, it is enough to find a cover of the final object of Y so that it pulls back to an isomorphism. Since \mathcal{F} is quasi-coherent, there is a cover of the final object of X so that the pullback of \mathcal{F} has a presentation. Pulling that cover back along f , we get a cover of the final object of Y (here we're using exactness of f^{-1} to say that the final object pulls back to the final object and that covers pull back to covers on canonical sites). On that cover, the map is an isomorphism by case 2. The construction of $\mathcal{H}om$, the application of f^* , and the construction of the natural map are local on X , so the natural morphism constructed on the cover is the restriction of the natural morphism on Y . \square

Lemma 7. *If \mathcal{X} is a Noetherian algebraic stack and $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}$ is a quasi-coherent sheaf of ideals, then $\widehat{\mathcal{O}}_{\mathcal{X}}$, the completion of $\mathcal{O}_{\mathcal{X}}$ with respect to \mathcal{I} , is flat over $\mathcal{O}_{\mathcal{X}}$. That is, the canonical map $\iota: \widehat{\mathcal{X}} \rightarrow \mathcal{X}$ is a flat morphism of ringed topoi.*

Proof. Let $\mathcal{F} \rightarrow \mathcal{F}'$ be an injection of $\mathcal{O}_{\mathcal{X}}$ -modules. We need to check injectivity of the map

$$\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \widehat{\mathcal{O}}_{\mathcal{X}} \rightarrow \mathcal{F}' \otimes_{\mathcal{O}_{\mathcal{X}}} \widehat{\mathcal{O}}_{\mathcal{X}}.$$

Since sheafification is exact, it suffices to check injectivity of the maps

$$\mathcal{F}(U) \otimes_{\mathcal{O}_{\mathcal{X}}(U)} \widehat{\mathcal{O}}_{\mathcal{X}}(U) \rightarrow \mathcal{F}'(U) \otimes_{\mathcal{O}_{\mathcal{X}}(U)} \widehat{\mathcal{O}}_{\mathcal{X}}(U)$$

as U varies over a base for $\mathcal{X}_{\text{lis-et}}$. Thus it suffices to check that the above maps are injections for $f: U \rightarrow \mathcal{X}$ a smooth map and U an affine scheme. By definition, $\mathcal{O}_{\mathcal{X}}(U) = \mathcal{O}_U(U)$ and $\widehat{\mathcal{O}}_{\mathcal{X}}(U) = \widehat{\mathcal{O}}_U(U)$. Since U is affine, $\widehat{\mathcal{O}}_U(U) = \widehat{\mathcal{O}_U(U)}$. Injectivity follows since $\widehat{\mathcal{O}_U(U)}$ is flat over $\mathcal{O}_U(U)$ [Eis95, Theorem 7.2b]. \square

Remark 8. The same trick of restricting to affine schemes smooth over \mathcal{X} shows that for any coherent sheaf \mathcal{F} on \mathcal{X} , the natural map $\iota^*\mathcal{F} \rightarrow \widehat{\mathcal{F}}$ is an isomorphism. (Note however that this is not true for *quasi-coherent* sheaves.)

Remark 9. Lemma 7 and Remark 8 show that completion of coherent sheaves is exact.

Lemma 10. *Let \mathcal{X} be a Noetherian algebraic stack, A a ring, and $\mathfrak{m} \subseteq A$ an ideal. Suppose $\phi: \mathcal{X} \rightarrow \text{Spec } A$ is cohomologically affine, and $0 \rightarrow \mathcal{F}'' \rightarrow \mathcal{F} \rightarrow \mathcal{F}' \rightarrow 0$ is an exact sequence of coherent $\mathcal{O}_{\mathcal{X}}$ -modules. Then the induced sequence $0 \rightarrow \Gamma(\widehat{\mathcal{F}}'') \rightarrow \Gamma(\widehat{\mathcal{F}}) \rightarrow \Gamma(\widehat{\mathcal{F}}') \rightarrow 0$ is exact.*

Remark 11. By Remark 9, completion is exact, and by cohomological affineness, ϕ_* (which we identify with Γ) is exact on the category of quasi-coherent \mathcal{O}_X -modules. However, this does not prove Lemma 10 because the completion of a quasi-coherent \mathcal{O}_X -module will typically not be quasi-coherent as an \mathcal{O}_X -module. Indeed, completion of a module does not commute with localization, so even the completion of a quasi-coherent sheaf on an *affine scheme* may fail to be quasi-coherent.

Proof of Lemma 10. For purposes of checking exactness, A -module structure is irrelevant, so we may identify Γ with ϕ_* . We denote by \mathcal{I} the sheaf of ideals in \mathcal{O}_X generated by $\phi^*(\mathfrak{m})$.

By Remark 9, the sequence $0 \rightarrow \widehat{\mathcal{F}}'' \rightarrow \widehat{\mathcal{F}} \rightarrow \widehat{\mathcal{F}}' \rightarrow 0$ is exact. Taking global sections is a left exact functor, so we only need to prove that $\Gamma(\widehat{\mathcal{F}}) \rightarrow \Gamma(\widehat{\mathcal{F}}')$ is surjective.

Since $\Gamma = \phi_*$ is right adjoint to ϕ^* , it commutes with projective limits, so $\Gamma(\widehat{\mathcal{G}}) = \varprojlim \Gamma(\mathcal{G}/\mathcal{I}^{n+1})$ for any \mathcal{O}_X -module \mathcal{G} . That is, an element of $\Gamma(\widehat{\mathcal{G}})$ can be identified with a compatible sequence of elements of $\Gamma(\mathcal{G}/\mathcal{I}^{n+1})$. To prove the lemma, it suffices to show that for any $r_{n-1} \in \Gamma(\mathcal{F}/\mathcal{I}^n \mathcal{F})$ and $s_n \in \Gamma(\mathcal{F}'/\mathcal{I}^{n+1} \mathcal{F}')$, both mapping to $s_{n-1} \in \Gamma(\mathcal{F}'/\mathcal{I}^n \mathcal{F}')$, there exists some $r_n \in \Gamma(\mathcal{F}/\mathcal{I}^{n+1} \mathcal{F})$ lifting both r_{n-1} and s_n . This follows from the diagram chase below.³ Tensoring with $\mathcal{O}_X/\mathcal{I}^n$ is right exact, and Γ is exact on quasi-coherent sheaves by cohomological affineness, so the rows in the following diagram are exact.

$$\begin{array}{ccccccc}
\Gamma(\mathcal{F}''/\mathcal{I}^{n+1} \mathcal{F}'') & \xrightarrow{f_n} & \Gamma(\mathcal{F}/\mathcal{I}^{n+1} \mathcal{F}) & \xrightarrow{g_n} & \Gamma(\mathcal{F}'/\mathcal{I}^{n+1} \mathcal{F}') & \longrightarrow & 0 \\
\downarrow p''_n & & \downarrow p_n & & \downarrow p'_n & & \\
\Gamma(\mathcal{F}''/\mathcal{I}^n \mathcal{F}'') & \xrightarrow{f_{n-1}} & \Gamma(\mathcal{F}/\mathcal{I}^n \mathcal{F}) & \xrightarrow{g_{n-1}} & \Gamma(\mathcal{F}'/\mathcal{I}^n \mathcal{F}') & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccc}
& & & & a & \xrightarrow{\quad} & s_n \\
& & r_n := a + e & \xrightarrow{\quad} & \downarrow & & \downarrow \\
d & \xrightarrow{\quad} & e & \xrightarrow{\quad} & b & \xrightarrow{\quad} & s_{n-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
c & \xrightarrow{\quad} & r_{n-1} - b & \xrightarrow{\quad} & 0 & &
\end{array}$$

By surjectivity of g_n , there is an a so that $g_n(a) = s_n$. Let $b = p_n(b)$. By commutativity of the right square, $g_{n-1}(b) = s_{n-1}$. By additivity of g_{n-1} , we have that $g_{n-1}(r_{n-1} - b) = 0$, so by exactness of the bottom row, there is a c so that $f_{n-1}(c) = r_{n-1} - b$.

By cohomological affineness of ϕ , the map $\Gamma(\mathcal{G}/\mathcal{I}^{n+1}) \rightarrow \Gamma(\mathcal{G}/\mathcal{I}^n \mathcal{G})$ is surjective for any quasi-coherent \mathcal{O}_X -module \mathcal{G} . In particular, p''_n is surjective, so there is some d so that $p''_n(d) = c$. Let $e = f_n(d)$, so $p_n(e) = r_{n-1} - b$. Then we may set $r_n = a + e$ since $g_n(a + e) = g_n(a) = s_n$ and $p_n(a + e) = b + (r_{n-1} - b) = r_{n-1}$. \square

Lemma 12. Suppose $\phi: \mathcal{X} \rightarrow \operatorname{Spec} A$ is a good moduli space, where A is a complete Noetherian local ring with maximal ideal \mathfrak{m} . If $\mathcal{F} \in \operatorname{Coh}(\mathcal{X})$, then the natural map $\Gamma(\mathcal{F}) \rightarrow \Gamma(\widehat{\mathcal{F}})$ is an isomorphism.

Proof. As in the proof of Lemma 10, we identify Γ with ϕ_* .

Case 1: First we show the result is true for $\mathcal{F} = \mathcal{O}_X$. We have that $\Gamma(\widehat{\mathcal{O}}_X) = \varprojlim \Gamma(\mathcal{O}_X/\mathcal{I}^n)$. Since $\mathcal{O}_X/\mathcal{I}^n$ is the structure sheaf of $\mathcal{X} \times_{\operatorname{Spec} A} \operatorname{Spec} A/\mathfrak{m}^n$ and since good moduli space maps are stable under base change [Alp09, Proposition 4.7(i)], we have that $\Gamma(\mathcal{O}_X/\mathcal{I}^n) = A/\mathfrak{m}^n$ (this is

³The objects of the lower diagram are elements of the corresponding groups in the upper diagram. A solid arrow between two objects indicates that one of them is defined or constructed from the other. A dotted arrow between two objects indicates relationship they must have (because of commutativity, additivity, exactness, or some other information).

the Stein hypothesis). Therefore, $\Gamma(\widehat{\mathcal{O}}_{\mathcal{X}}) = \widehat{A} = A$, as desired. The result follows for $\mathcal{F} = \mathcal{O}_{\mathcal{X}}^{\oplus m}$ for any finite m since Γ commutes with finite direct sums.

Case 2: Suppose $\Gamma(\mathcal{F}) = 0$. As usual, we have $\Gamma(\widehat{\mathcal{F}}) = \varprojlim \Gamma(\mathcal{F}/\mathcal{I}^n \mathcal{F})$. Since ϕ is cohomologically affine, $\Gamma(\mathcal{F}) \rightarrow \Gamma(\mathcal{F}/\mathcal{I}^n \mathcal{F})$ is surjective, so if \mathcal{F} has no global sections, neither does $\mathcal{F}/\mathcal{I}^n \mathcal{F}$, so neither does $\widehat{\mathcal{F}}$.

Case 3: Now suppose \mathcal{F} is an arbitrary coherent sheaf. Let \mathcal{F}' be the sub- $\mathcal{O}_{\mathcal{X}}$ -module generated by the global sections of \mathcal{F} . By [Alp09, Theorem 4.16(x)], $\Gamma(\mathcal{F})$ is a finitely generated A -module, so there exists a surjection $\mathcal{O}_{\mathcal{X}}^m \rightarrow \mathcal{F}'$ with m finite. Let $\mathcal{G} = \ker(\mathcal{O}_{\mathcal{X}}^m \rightarrow \mathcal{F})$, and let \mathcal{G}' be the sub- $\mathcal{O}_{\mathcal{X}}$ -module generated by the global sections of \mathcal{G} . Again, as $\Gamma(\mathcal{G})$ is finitely generated, there is a surjection $\mathcal{O}_{\mathcal{X}}^k \rightarrow \mathcal{G}'$ with k finite. Consider the following diagram composed out of short exact sequences:

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & \nearrow & \\
 & 0 & & \mathcal{G}/\mathcal{G}' & \nearrow 0 \\
 & \searrow & \nearrow & \searrow & \\
 \mathcal{O}_{\mathcal{X}}^k & \rightarrow & \mathcal{G} & \rightarrow & \mathcal{O}_{\mathcal{X}}^m \\
 \searrow & \nearrow & \searrow & \nearrow & \\
 0 & \rightarrow & \mathcal{G}' & \rightarrow & 0 \\
 & & \searrow & & \\
 & & \mathcal{F}' & \rightarrow & 0 \\
 & \nearrow & \nearrow & \nearrow & \\
 & 0 & \rightarrow & \mathcal{F} & \rightarrow 0 \\
 & & \searrow & \nearrow & \\
 & & \mathcal{F}/\mathcal{F}' & \rightarrow & 0
 \end{array} \tag{*}$$

Applying Γ to (*), the exact sequences remain exact by cohomological affineness of ϕ . We therefore have that $\Gamma(\mathcal{F}/\mathcal{F}') \cong \Gamma(\mathcal{F})/\Gamma(\mathcal{F}') = 0$ and $\Gamma(\mathcal{G}/\mathcal{G}') \cong \Gamma(\mathcal{G})/\Gamma(\mathcal{G}') = 0$, so the induced sequence $\Gamma(\mathcal{O}_{\mathcal{X}}^k) \rightarrow \Gamma(\mathcal{O}_{\mathcal{X}}^m) \rightarrow \Gamma(\mathcal{F}) \rightarrow 0$ is exact. On the other hand, we can first complete everything in (*) and then take global sections. The sequences remain exact by Lemma 10. By case 2, $\Gamma(\widehat{\mathcal{F}'}/\widehat{\mathcal{F}}) = 0$ and $\Gamma(\widehat{\mathcal{G}'}/\widehat{\mathcal{G}}) = 0$, so the induced sequence $\Gamma(\widehat{\mathcal{O}}_{\mathcal{X}}^k) \rightarrow \Gamma(\widehat{\mathcal{O}}_{\mathcal{X}}^m) \rightarrow \Gamma(\widehat{\mathcal{F}}) \rightarrow 0$ is exact. We now have the following commutative diagram with exact rows. The squares commute by naturality of the vertical arrows.

$$\begin{array}{ccccccc}
 \Gamma(\mathcal{O}_{\mathcal{X}}^k) & \longrightarrow & \Gamma(\mathcal{O}_{\mathcal{X}}^m) & \longrightarrow & \Gamma(\mathcal{F}) & \longrightarrow & 0 \\
 \downarrow \wr & & \downarrow \wr & & \downarrow & & \\
 \Gamma(\widehat{\mathcal{O}}_{\mathcal{X}}^k) & \longrightarrow & \Gamma(\widehat{\mathcal{O}}_{\mathcal{X}}^m) & \longrightarrow & \Gamma(\widehat{\mathcal{F}}) & \longrightarrow & 0.
 \end{array}$$

The left and middle vertical arrows are isomorphisms by case 1, so $\Gamma(\mathcal{F}) \rightarrow \Gamma(\widehat{\mathcal{F}})$ is also an isomorphism. \square

Lemma 13. *Any vector bundle \mathcal{V} on \mathcal{X}_{n-1} is the reduction of a unique vector bundle on \mathcal{X}_n . In particular, any vector bundle on \mathcal{X}_0 extends to a unique vector bundle on $\widehat{\mathcal{X}}$.*

Proof. This is a direct application of [FGI⁺05, Theorem 8.5.3(b)]. The obstruction to extending \mathcal{V} to \mathcal{X}_n lies in $H^2(\mathcal{X}_{n-1}, \mathcal{I}^n \otimes \mathcal{E}nd(\mathcal{V}))$, which vanishes since \mathcal{X}_n is cohomologically affine. Therefore \mathcal{V} extends. Moreover, the isomorphism classes of extensions are parameterized by $H^1(\mathcal{X}_{n-1}, \mathcal{I}^n \otimes \mathcal{E}nd(\mathcal{V}))$, which vanishes by the same argument, so the extension is unique. \square

Lemma 14. *A quasi-coherent sheaf \mathcal{F} on a locally Noetherian stack \mathcal{X} is a flat $\mathcal{O}_{\mathcal{X}}$ -module (i.e. restricts to a flat sheaf on any smooth cover by a scheme) if and only if $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} -$ is an exact functor on $\text{QCoh}(\mathcal{X})$.*

Proof. Suppose \mathcal{F} is flat and $\mathcal{G} \rightarrow \mathcal{G}'$ is an injection of quasi-coherent sheaves. Let $f: U \rightarrow \mathcal{X}$ be a smooth cover by a scheme. We may check that $\mathcal{F} \otimes \mathcal{G} \rightarrow \mathcal{F} \otimes \mathcal{G}'$ is injective after pulling back to

U . Pullback respects tensor products, $f^*\mathcal{G} \rightarrow f^*\mathcal{G}'$ is injective (since f is flat), and $f^*\mathcal{F}$ is a flat \mathcal{O}_U -module, so $f^*(\mathcal{F} \otimes \mathcal{G}) \rightarrow f^*(\mathcal{F} \otimes \mathcal{G}')$ is injective.

For the converse, again let $f: U \rightarrow \mathcal{X}$ be a smooth cover by a scheme. We wish to prove $f^*\mathcal{F}$ is flat. This may be done locally on U , so we may assume U is a Noetherian affine scheme. The result is well-known for schemes, so it suffices to prove that $f^*\mathcal{F} \otimes_{\mathcal{O}_U} -$ is an exact functor on $\mathrm{QCoh}(U)$. First we claim that for any sheaf of \mathcal{O}_U -modules \mathcal{G} , the counit of adjunction $f^*f_*\mathcal{G} \rightarrow \mathcal{G}$ has a natural section. Indeed, let $W \rightarrow U$ be a smooth morphism. Then the map

$$f^*f_*\mathcal{G}(W \rightarrow U) \cong \mathcal{G}(U \times_{\mathcal{X}} W \rightarrow U) \rightarrow \mathcal{G}(W \rightarrow U)$$

has a section given by the restriction map

$$\mathcal{G}(W \rightarrow U) \rightarrow \mathcal{G}(U \times_{\mathcal{X}} W \rightarrow W \rightarrow U).$$

Now let $\mathcal{G} \rightarrow \mathcal{G}'$ be an injection of quasi-coherent sheaves on U . Since U is Noetherian, f is quasi-compact and quasi-separated, so $f_*\mathcal{G} \rightarrow f_*\mathcal{G}'$ is an injection of quasi-coherent $\mathcal{O}_{\mathcal{X}}$ -modules. By assumption, $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} f_*\mathcal{G} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} f_*\mathcal{G}'$ is an injection, and since f is flat, $\phi: f^*(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} f_*\mathcal{G}) \rightarrow f^*(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} f_*\mathcal{G}')$ is an injection. Noting that $f^*(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} f_*(-)) \cong f^*\mathcal{F} \otimes_{\mathcal{O}_U} f^*f_*(-)$, we get a diagram

$$\begin{array}{ccc} f^*\mathcal{F} \otimes_{\mathcal{O}_U} f^*f_*\mathcal{G} & \xrightarrow{\phi} & f^*\mathcal{F} \otimes_{\mathcal{O}_U} f^*f_*\mathcal{G}' \\ \sigma \uparrow \downarrow \pi & & \sigma' \uparrow \downarrow \\ f^*\mathcal{F} \otimes_{\mathcal{O}_U} \mathcal{G} & \xrightarrow{\psi} & f^*\mathcal{F} \otimes_{\mathcal{O}_U} \mathcal{G}' \end{array}$$

We have that ϕ is injective, and σ is injective (since it is a section of π). Since $\sigma' \circ \psi = \phi \circ \sigma$, we conclude that ψ is injective. \square

Proof of Theorem 1. Part (i): For any coherent $\mathcal{O}_{\mathcal{X}}$ -modules \mathcal{F} and \mathcal{G} , we must show that the natural map $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) \rightarrow \mathrm{Hom}_{\widehat{\mathcal{O}_{\mathcal{X}}}}(\widehat{\mathcal{F}}, \widehat{\mathcal{G}})$ is an isomorphism. We have that $\widehat{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is coherent. By Lemma 6, Lemma 7, and Remark 8, the natural map $\widehat{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) \rightarrow \widehat{\mathrm{Hom}}_{\widehat{\mathcal{O}_{\mathcal{X}}}}(\widehat{\mathcal{F}}, \widehat{\mathcal{G}})$ is an isomorphism. By Lemma 12, the induced map on global sections is the desired isomorphism.

(res) \Rightarrow (res'). This is immediate since any coherent sheaf on \mathcal{X}_0 is a coherent sheaf on \mathcal{X} .

(res') \Rightarrow (GAGA). By part (i), the completion functor is fully faithful. It remains to show that any compatible system $\mathfrak{F} = \{\mathcal{F}_n\}_{n \geq 0}$ of coherent sheaves on the stacks \mathcal{X}_n is induced by a coherent sheaf \mathcal{F} on \mathcal{X} . As usual, we denote by \mathcal{I} the quasi-coherent sheaf of ideals generated by $\phi^*(\mathfrak{m})$.

First, we show that for any coherent sheaf \mathcal{F}_n on \mathcal{X}_n , there exists a locally free sheaf \mathcal{V} on \mathcal{X} and a surjection $\mathcal{V} \rightarrow \mathcal{F}_n$. By (res'), this is true for $n = 0$. The bottom row of the following diagram is exact:

$$\begin{array}{ccccccc} & & \mathcal{V}' & & & \mathcal{V}'' & \\ & & \downarrow & & \swarrow & \downarrow & \\ 0 & \longrightarrow & \mathcal{I}^n \mathcal{F}_n & \longrightarrow & \mathcal{F}_n & \longrightarrow & \mathcal{F}_n / \mathcal{I}^n \mathcal{F}_n \longrightarrow 0 \end{array} \quad (\ddagger)$$

Since $\mathcal{I}^n \mathcal{F}_n$ is a coherent sheaf supported on \mathcal{X}_0 , there is a surjection from a vector bundle \mathcal{V}' . Since $\mathcal{F}_n / \mathcal{I}^n \mathcal{F}_n$ is a coherent sheaf supported on \mathcal{X}_{n-1} , there is a surjection from a vector bundle \mathcal{V}'' by induction. Since \mathcal{V}'' is a vector bundle, the following sequence is exact:

$$0 \rightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{V}'', \mathcal{I}^n \mathcal{F}_n) \rightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{V}'', \mathcal{F}_n) \rightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{V}'', \mathcal{F}_n / \mathcal{I}^n \mathcal{F}_n) \rightarrow 0.$$

By cohomological affineness of ϕ , the sequence remains exact when we take global sections, so the composition map $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{V}'', \mathcal{F}_n) \rightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{V}'', \mathcal{F}_n / \mathcal{I}^n \mathcal{F}_n)$ is surjective. Thus, there is a lift $\mathcal{V}'' \rightarrow \mathcal{F}_n$ as indicated by the dotted arrow in (\ddagger) . The induced map $\mathcal{V}' \oplus \mathcal{V}'' \rightarrow \mathcal{F}_n$ is a surjection.

Next, we show that \mathfrak{F} has a surjection from a vector bundle \mathcal{V} on \mathcal{X} (i.e. there is a compatible family of surjections $\mathcal{V} \rightarrow \mathcal{F}_n$). For each n , we choose a surjection $\mathcal{V}^n \rightarrow \mathcal{F}_n$, where \mathcal{V}^n is a vector bundle on \mathcal{X} (not on \mathcal{X}_n) and \mathcal{F}_n is regarded as a coherent sheaf on \mathcal{X} . For any m , since $\mathcal{F}_{m+1} \rightarrow \mathcal{F}_m$ is surjective and \mathcal{V}^n is a vector bundle, $\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{V}^n, \mathcal{F}_{m+1}) \rightarrow \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{V}^n, \mathcal{F}_m)$ is surjective, so by cohomological affineness of ϕ , $\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{V}^n, \mathcal{F}_{m+1}) \rightarrow \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{V}^n, \mathcal{F}_m)$ is surjective. Thus, the surjection $\mathcal{V}^n \rightarrow \mathcal{F}_n$ can be extended to a compatible system of maps $\{\mathcal{V}^n \rightarrow \mathcal{F}_m\}_{m \geq 0}$, so we get a morphism $\widehat{\mathcal{V}}^n \rightarrow \mathfrak{F}$ which is surjective modulo \mathcal{I}^{n+1} . The images of $\bigoplus_{n=0}^N \widehat{\mathcal{V}}^n$ form an increasing sequence of coherent $\widehat{\mathcal{O}}_{\mathcal{X}}$ -submodules of \mathfrak{F} . Since $\mathcal{O}_{\mathcal{X}}$ is Noetherian, so is $\widehat{\mathcal{O}}_{\mathcal{X}}$, so there is some finite sum $\mathcal{V} = \bigoplus_{n=0}^N \mathcal{V}^n$ so that the image of $\widehat{\mathcal{V}} \rightarrow \mathfrak{F}$ agrees with the image of $\bigoplus_{n=0}^{\infty} \widehat{\mathcal{V}}^n \rightarrow \mathfrak{F}$. Then $\mathcal{V} \rightarrow \mathcal{F}_n$ is surjective for each n , so $\widehat{\mathcal{V}} \rightarrow \mathfrak{F}$ is surjective.

Repeating the above argument for the kernel of $\widehat{\mathcal{V}} \rightarrow \mathfrak{F}$, we get a presentation $\widehat{\mathcal{W}} \rightarrow \widehat{\mathcal{V}} \rightarrow \mathfrak{F} \rightarrow 0$, where \mathcal{V} and \mathcal{W} are vector bundles on \mathcal{X} . By part (i), the morphism $\widehat{\mathcal{W}} \rightarrow \widehat{\mathcal{V}}$ is induced by some $\mathcal{O}_{\mathcal{X}}$ -module homomorphism $\mathcal{W} \rightarrow \mathcal{V}$. Let \mathcal{G} be the cokernel of this map. By Remark 9, the top row of the following diagram is exact.

$$\begin{array}{ccccccc} \widehat{\mathcal{W}} & \longrightarrow & \widehat{\mathcal{V}} & \longrightarrow & \widehat{\mathcal{G}} & \longrightarrow & 0 \\ \parallel & & \parallel & & \vdots & & \\ \widehat{\mathcal{W}} & \longrightarrow & \widehat{\mathcal{V}} & \longrightarrow & \mathfrak{F} & \longrightarrow & 0 \end{array}$$

The induced morphism from $\widehat{\mathcal{G}}$ to \mathfrak{F} is therefore an isomorphism.

(res') \Rightarrow (res). The above argument shows that if (res') holds and \mathcal{F} is a coherent sheaf on \mathcal{X} , then there is a vector bundle \mathcal{V} on \mathcal{X} and a surjection $\widehat{\mathcal{V}} \rightarrow \widehat{\mathcal{F}}$. Since (res') \Rightarrow (GAGA), this map is induced by a surjection $\mathcal{V} \rightarrow \mathcal{F}$.

(GAGA) \Rightarrow (res'). First we show that if (GAGA) holds and $\mathcal{F} \in \text{Coh}(\mathcal{X})$ completes to a vector bundle on $\widehat{\mathcal{X}}$, then \mathcal{F} is a vector bundle. By Remark 8, the equivalence of categories of coherent sheaves respects tensor products, so since $\widehat{\mathcal{F}} \otimes_{\widehat{\mathcal{O}}_{\mathcal{X}}} -$ is an exact functor on $\text{Coh}(\widehat{\mathcal{X}})$, we have that $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} -$ is an exact functor on $\text{Coh}(\mathcal{X})$. Let $\text{Spec } R \rightarrow \mathcal{X}$ be a smooth cover of \mathcal{X} (note \mathcal{X} is assumed finite type over the Noetherian ring A , so it is quasi-compact). Then $\text{Spec } R \times_{\mathcal{X}} \text{Spec } R$ is of finite type over $\text{Spec } A$, so the projections $\text{Spec } R \times_{\mathcal{X}} \text{Spec } R \rightarrow \text{Spec } R$ are smooth, quasi-compact, and quasi-separated, so any quasi-coherent sheaf on \mathcal{X} is the limit of its coherent subsheaves [Stacks, Lemma 07TU]. Since $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} -$ commutes with direct limits, it is exact on the category of *quasi*-coherent sheaves, so \mathcal{F} is a flat $\mathcal{O}_{\mathcal{X}}$ -module by Lemma 14. It follows that \mathcal{F} is a vector bundle; indeed, this can be checked smooth locally, and a flat coherent sheaf on a Noetherian affine scheme is locally free [Stacks, Lemma 00NX].

Now for any coherent sheaf \mathcal{F}_0 on \mathcal{X}_0 , since \mathcal{X}_0 is assumed to have the resolution property, there is a vector bundle \mathcal{V}_0 on \mathcal{X}_0 with a surjection to \mathcal{F}_0 . By Lemma 13, \mathcal{V}_0 extends to a vector bundle on $\widehat{\mathcal{X}}$, which by formal GAGA is the completion of a coherent sheaf \mathcal{V} on \mathcal{X} . By the above paragraph, \mathcal{V} is a vector bundle. Now $\mathcal{V} \rightarrow \mathcal{V}_0 \rightarrow \mathcal{F}_0$ is a surjection. This shows that (res') holds.

(quot) \Leftrightarrow (quot'). It is clear that (quot) \Rightarrow (quot'). Conversely, suppose $\mathcal{X} = [V/G]$ for some algebraic space V and some subgroup $G \subseteq GL_n$. Letting $U = (V \times GL_n)/G$, where $g \cdot (v, h) = (v \cdot g^{-1}, g \cdot h)$,⁴ we have that $\mathcal{X} = [U/GL_n]$. Then $U \rightarrow \mathcal{X}$ is a GL_n -torsor, so an affine morphism, and $\mathcal{X} \rightarrow \text{Spec } A$ is cohomologically affine, so $U \rightarrow \text{Spec } A$ is cohomologically affine. As U has trivial stabilizers, it is an algebraic space, so by Serre's criterion [Knu71, Theorem III.2.5], U is an affine scheme.

⁴Alternatively, U is the pullback of the universal GL_n -torsor along the composition $[V/G] \rightarrow BG \rightarrow BGL_n$.

(quot) \Rightarrow (res) \Rightarrow (quot'). By [Tot04, Theorem 1.1], (quot) \Rightarrow (res), and if the closed point of \mathcal{X} has affine stabilizer, then (res) \Rightarrow (quot'). \square

Remark 15. The proofs of (quot) \Leftrightarrow (quot') and (quot) \Rightarrow (res) \Rightarrow (quot') apply to *any* stack with affine good moduli space. Note however that (res) \Rightarrow (quot') requires all closed points of the stack to have affine stabilizer.

Remark 16. Note that the proof of (GAGA) \Rightarrow (res') shows that (GAGA) implies that any coherent sheaf whose completion is a vector bundle must be a vector bundle. The hypothesis that \mathcal{X}_0 have the resolution property is not necessary for this result.

Remark 17. Suppose $A/\mathfrak{m} = k$ and A is a k -algebra.⁵ If $\mathcal{X} \cong \mathcal{X}_0 \times_{\mathrm{Spec} k} \mathrm{Spec} A$, then we have a morphism $s: \mathcal{X} \rightarrow \mathcal{X}_0$ so that $\mathcal{X}_0 \hookrightarrow \mathcal{X} \xrightarrow{s} \mathcal{X}_0$ is the identity map. Any vector bundle $\mathcal{V}_0 \in \mathrm{Coh}(\mathcal{X}_0)$ is the reduction of the vector bundle $s^*\mathcal{V}_0 \in \mathrm{Coh}(\mathcal{X})$. If \mathcal{X}_0 has the resolution property, then any $\mathcal{F}_0 \in \mathrm{Coh}(\mathcal{X}_0)$ has a surjection from a vector bundle $\mathcal{V}_0 \in \mathrm{Coh}(\mathcal{X}_0)$, so the map $s^*\mathcal{V}_0 \rightarrow \mathcal{V}_0 \rightarrow \mathcal{F}_0$ is a surjection from a vector bundle on \mathcal{X} . That is, if \mathcal{X}_0 has the resolution property, (res') holds.

Note however that the condition $\mathcal{X} \cong \mathcal{X}_0 \times_{\mathrm{Spec} k} \mathrm{Spec} A$ is frequently not satisfied. For example, consider the j -invariant map $j: \mathcal{M}_{1,1} \rightarrow \mathbb{A}_{\mathbb{C}}^1$ and let $\mathcal{X} \rightarrow \mathrm{Spec} \mathbb{C}[[t]]$ be the pullback of j to the local ring of $\mathbb{A}_{\mathbb{C}}^1$ at the origin. Since elliptic curves with j -invariant 0 have automorphism group $\mathbb{Z}/4\mathbb{Z}$ but generic elliptic curves have automorphism group $\mathbb{Z}/2\mathbb{Z}$, \mathcal{X} cannot be the pullback of its special fiber.

3. FORMAL GAGA IS ÉTALE LOCAL ON THE BASE

Lemma 18. *Suppose $\phi: \mathcal{X} \rightarrow \mathrm{Spec} A$ is a good moduli space, where A is a complete Noetherian local ring and ϕ is of finite type. Suppose $\mathrm{Spec} A' \rightarrow \mathrm{Spec} A$ is a finite étale morphism, where A' is again local (and therefore a complete Noetherian local ring), and let $\mathcal{X}' = \mathcal{X} \times_{\mathrm{Spec} A} \mathrm{Spec} A'$.*

If $\mathrm{Coh}(\mathcal{X}') \rightarrow \mathrm{Coh}(\widehat{\mathcal{X}}')$ is essentially surjective, then so is $\mathrm{Coh}(\mathcal{X}) \rightarrow \mathrm{Coh}(\widehat{\mathcal{X}})$.

Remark 19. By Remark 8, completion of coherent sheaves agrees with pullback along the morphism of topoi $\iota: \widehat{\mathcal{X}} \rightarrow \mathcal{X}$. It follows that pullback along $\pi: \mathcal{X}' \rightarrow \mathcal{X}$ commutes with completion of coherent sheaves, and that completion of coherent sheaves is a right exact functor.

Proof of Lemma 18. Good moduli space morphisms are stable under base change [Alp09, Proposition 4.7(i)], so $\mathcal{X}'' = \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}' \rightarrow \mathrm{Spec} A'' = \mathrm{Spec}(A' \otimes_A A')$ is a good moduli space. Let $p_1, p_2: \mathcal{X}'' \rightarrow \mathcal{X}'$ denote the projections. While A'' may no longer be a local ring, $\mathrm{Spec} A''$ is finite étale over $\mathrm{Spec} A'$, so it must be a disjoint union $\bigsqcup \mathrm{Spec} A_i''$, where each A_i'' is a complete local ring. Let $\mathcal{X}_i'' = \mathcal{X}'' \times_{\mathrm{Spec} A''} \mathrm{Spec} A_i''$.

Let $\mathfrak{F} \in \mathrm{Coh}(\widehat{\mathcal{X}})$, and let $\mathfrak{F}' \in \mathrm{Coh}(\widehat{\mathcal{X}}')$ be the pullback to \mathcal{X}' . By assumption, \mathfrak{F}' is the completion of a sheaf $\mathcal{F}' \in \mathrm{Coh}(\mathcal{X}')$. Applying Theorem 1(i) to each of the good moduli space morphisms $\mathcal{X}_i'' \rightarrow \mathrm{Spec} A_i''$, we see that the descent data $p_2^*\mathfrak{F}' \xrightarrow{\sim} p_1^*\mathfrak{F}'$ is induced by a map $p_2^*\mathcal{F}' \xrightarrow{\sim} p_1^*\mathcal{F}'$ (note we are using Remark 19). It follows that \mathcal{F}' is the pullback of a coherent sheaf \mathcal{F} on \mathcal{X} . Since $\widehat{\mathcal{F}}$ and \mathfrak{F} are defined by the same descent data, they are isomorphic. \square

Theorem 20 (Formal GAGA is étale local on the base). *In the setup of Lemma 18, formal GAGA holds for $\mathcal{X} \rightarrow \mathrm{Spec} A$ if and only if it holds for $\mathcal{X}' \rightarrow \mathrm{Spec} A'$.*

Proof. By Theorem 1(i), both completion functors are fully faithful.

⁵This is automatic if k has characteristic zero. Every non-negative integer is non-zero in k , so lies in $A \setminus \mathfrak{m}$, so is invertible in A . This shows that A is a \mathbb{Q} -algebra. By [Eis95, Theorem 7.7], it is a k -algebra.

By Lemma 18, if the completion functor $\mathrm{Coh}(\mathcal{X}') \rightarrow \mathrm{Coh}(\widehat{\mathcal{X}}')$ is essentially surjective, then so is $\mathrm{Coh}(\mathcal{X}) \rightarrow \mathrm{Coh}(\widehat{\mathcal{X}})$.

Conversely, suppose $\mathrm{Coh}(\mathcal{X}) \rightarrow \mathrm{Coh}(\widehat{\mathcal{X}})$ is essentially surjective, and let $\mathfrak{F} \in \mathrm{Coh}(\widehat{\mathcal{X}}')$. Since $\pi: \mathcal{X}' \rightarrow \mathcal{X}$ is finite, $\pi_*\mathfrak{F} \in \mathrm{Coh}(\widehat{\mathcal{X}})$. By assumption, $\pi_*\mathfrak{F} \cong \widehat{\mathcal{F}}$ for some $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$. The composition $\pi^*\widehat{\mathcal{F}} \xrightarrow{\sim} \pi^*\pi_*\mathfrak{F} \rightarrow \mathfrak{F}$ is a surjection. Let \mathfrak{G} denote the kernel of this map. By the same argument, there exists a surjection $\pi^*\widehat{\mathcal{G}} \rightarrow \mathfrak{G}$ for some $\mathcal{G} \in \mathrm{Coh}(\mathcal{X})$. Then \mathfrak{F} is the cokernel of the map $\pi^*\widehat{\mathcal{G}} \rightarrow \pi^*\widehat{\mathcal{F}}$. By full faithfulness and Remark 19, this map is induced by a morphism $\pi^*\mathcal{G} \rightarrow \pi^*\mathcal{F}$, and the cokernel of this map has completion \mathfrak{F} . \square

4. COUNTEREXAMPLES TO FORMAL GAGA

Recall that for a relative group $G \rightarrow S$, a coherent sheaf on $BG = [S/G]$ is equivalent to a coherent sheaf on S with a G -linearization (i.e. a G -action). Pushforward along $\phi: BG \rightarrow S$ corresponds to taking the subsheaf of invariants; in particular, since \mathcal{O}_{BG} corresponds to \mathcal{O}_S with the trivial G -action, ϕ is Stein. Since the action of G on S is trivial, ϕ is universal for maps to algebraic spaces.⁶ The condition that the map be cohomologically affine is precisely the condition that G is linearly reductive. Therefore $BG \rightarrow S$ is a good moduli space if and only if G is linearly reductive.

Formal GAGA fails without the good moduli space condition. In the following, we say that a morphism to an algebraic space $\mathcal{X} \rightarrow X$ is a *no-good moduli space* if it is universal for maps to algebraic spaces but is not a good moduli space.

Example 21 (Counterexample to full faithfulness for a no-good moduli space). Let $A = k[[t]]$ for a field k . Let $G = \mathrm{Spec} k[[t]] \sqcup \mathrm{Spec} k((t))$, regarded as an open subgroup of $(\mathbb{Z}/2)_{\mathrm{Spec} A}$. Then $\mathcal{X} = BG \rightarrow \mathrm{Spec} A$ is not a good moduli space. The non-trivial 1-dimensional representation of $\mathbb{Z}/2$ induces a non-trivial rank 1 vector bundle on \mathcal{X} whose completion is the trivial rank 1 vector bundle on $\widehat{\mathcal{X}}$ (indeed, $\widehat{\mathcal{X}} \cong \mathrm{Spec} A$), showing that the completion functor is not fully faithful. \diamond

Example 22 (Counterexample to essential surjectivity for a no-good moduli space). Formal GAGA fails for $B\mathbb{G}_a$. For a ring R , a line bundle on $B\mathbb{G}_{a,R}$ is equivalent to a 1-dimensional representation of $\mathbb{G}_{a,R}$ (i.e. a group homomorphism $\mathbb{G}_{a,R} \rightarrow \mathbb{G}_{m,R}$). The formula $x \mapsto \exp(tx) = \sum_{i=0}^{\infty} \frac{t^i}{i!} x^i$ gives a compatible family of homomorphisms $\mathbb{G}_{a,\mathbb{C}[[t]]/t^n} \rightarrow \mathbb{G}_{m,\mathbb{C}[[t]]/t^n}$ which do not lift to a homomorphism $\mathbb{G}_{a,\mathbb{C}[[t]]} \rightarrow \mathbb{G}_{m,\mathbb{C}[[t]]}$. \diamond

Formal GAGA may also fail for good moduli spaces.

Example 23 (Counterexample to essential surjectivity with non-separated diagonal). Let $A = k[[t]]$ for a field k . Let G be $\mathrm{Spec} k[[t]]$ with a doubled origin, regarded as a group over $\mathrm{Spec} A$. Since G is a quotient of $(\mathbb{Z}/2\mathbb{Z})_{\mathrm{Spec} A}$, it is linearly reductive, so $\mathcal{X} = BG \rightarrow \mathrm{Spec} A$ is a good moduli space.

Any vector bundle on \mathcal{X} consists of a vector bundle \mathcal{V} on $\mathrm{Spec} A$ and a group homomorphism $G_A \rightarrow \mathrm{Aut}_A(\mathcal{V})$. Since $\mathrm{Aut}_A(\mathcal{V})$ is separated, such a map must factor through the trivial group. So any vector bundle on \mathcal{X} corresponds to a vector bundle on $\mathrm{Spec} A$ with trivial G -action. However, $\widehat{\mathcal{X}} \cong B_{\mathrm{Spec} A}(\mathbb{Z}/2)$, so there are formal vector bundles not of this form, namely those induced by non-trivial representations of $\mathbb{Z}/2$. \diamond

Even if we require separated diagonal, formal GAGA may still fail.

Example 24 (Counterexample to essential surjectivity with separated, non-affine diagonal). Let

$$G' = \mathrm{Proj}(k[[t]][x, y, z]/(zy^2 - x^2(x + z) - tz^3))$$

⁶More generally, if $\alpha: G \times X \rightarrow X$ is an action of G on an algebraic space X and the two maps $\alpha, p_2: G \times X \rightarrow X$ have coequalizer Y in the category of algebraic spaces, then $[X/G] \rightarrow Y$ is universal for maps to algebraic spaces.

where t has degree 0 and x , y , and z have degree 1. Let G be the complement of the origin of the special fiber, with structure map $\pi: G \rightarrow \operatorname{Spec} k[[t]]$. The generic fiber is an elliptic curve $E \rightarrow \operatorname{Spec} k((t))$, but the special fiber is isomorphic to \mathbb{G}_m . By [Sil94, IV Theorem 5.3(c)], G is a relative group scheme over $\operatorname{Spec} k[[t]]$. We claim that $BG \rightarrow \operatorname{Spec} k[[t]]$ is a good moduli space morphism (i.e. that taking G -invariants is exact on G -linearized coherent sheaves).

To see this, we first note that any deformation of the group scheme \mathbb{G}_m is trivial. By [SGA3, Exposé III, Corollaire 3.9], isomorphism classes of deformations of the group scheme along a square-zero ideal I (if they exist) are parameterized by $H^2(\mathbb{G}_m, \operatorname{Lie}(\mathbb{G}_m) \otimes I)$, where $\operatorname{Lie}(\mathbb{G}_m)$ is the adjoint representation and I has the trivial action. The group cohomology $H^i(\mathbb{G}_m, -)$ as defined in [SGA3, Exposé III, 1.1] is simply the Čech cohomology associated to the cover $\operatorname{Spec} k[[t]] \rightarrow B\mathbb{G}_m$. Since \mathbb{G}_m is affine, this Čech cohomology agrees with sheaf cohomology on $B\mathbb{G}_m$. Since \mathbb{G}_m is linearly reductive, $B\mathbb{G}_m \rightarrow \operatorname{Spec} k[[t]]$ is cohomologically affine, so the higher cohomology groups vanish. Thus, the only deformation of \mathbb{G}_m is \mathbb{G}_m .

Next, any torsion G -linearized coherent sheaf is supported over $\operatorname{Spec}(k[t]/t^n)$ for some n . That is, there is some choice of n so that the given sheaf is in the essential image of j_* in the diagram below.

$$\begin{array}{ccc} (B\mathbb{G}_m)_{\operatorname{Spec}(k[t]/t^n)} \cong BG \times_{\operatorname{Spec} k[[t]]} \operatorname{Spec}(k[t]/t^n) & \xrightarrow{j} & BG \\ \pi_n \downarrow & & \downarrow \pi \\ \operatorname{Spec}(k[t]/t^n) & \xrightarrow{i} & \operatorname{Spec} k[[t]] \end{array}$$

Since i and j are affine, and π_n is cohomologically affine, we have

$$R\pi_* \circ j_* = R(\pi_* \circ j_*) = R(\pi_{n*} \circ i_*) = \pi_{n*} \circ i_* = \pi_* \circ j_*.$$

That is, torsion sheaves on BG have trivial higher cohomology.

Any torsion-free G -linearized coherent sheaf is free with trivial action. Indeed, it is free with some rank r since $k[[t]]$ is a DVR. The action of G is given by some group homomorphism $G \rightarrow GL_{r,k[[t]]}$. Since G has proper connected generic fiber and GL_r is affine, this map must be trivial over the generic point. Since G is reduced and GL_r is separated, the map must be trivial.

Any G -linearized coherent sheaf \mathcal{F} fits into a G -equivariant short exact sequence

$$0 \rightarrow \mathcal{F}^{\operatorname{tor}} \rightarrow \mathcal{F} \rightarrow \mathcal{F}/\mathcal{F}^{\operatorname{tor}} \rightarrow 0. \quad (*)$$

Since $\mathcal{F}/\mathcal{F}^{\operatorname{tor}}$ is free, the following sequence is exact.

$$0 \rightarrow \mathcal{H}om(\mathcal{F}/\mathcal{F}^{\operatorname{tor}}, \mathcal{F}^{\operatorname{tor}}) \rightarrow \mathcal{H}om(\mathcal{F}/\mathcal{F}^{\operatorname{tor}}, \mathcal{F}) \rightarrow \mathcal{H}om(\mathcal{F}/\mathcal{F}^{\operatorname{tor}}, \mathcal{F}/\mathcal{F}^{\operatorname{tor}}) \rightarrow 0.$$

Since $\mathcal{H}om(\mathcal{F}/\mathcal{F}^{\operatorname{tor}}, \mathcal{F}^{\operatorname{tor}})$ is torsion, $H^1(BG, \mathcal{H}om(\mathcal{F}/\mathcal{F}^{\operatorname{tor}}, \mathcal{F}^{\operatorname{tor}})) = 0$, so the sequence remains exact when we take global sections. Since global sections of $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is the group of G -equivariant maps from \mathcal{F} to \mathcal{G} , we conclude that there is a G -equivariant splitting of the sequence (*). We've shown that any G -linearized coherent $k[[t]]$ -module M decomposes into a direct sum of its torsion part M^{tor} (with trivial cohomology) and a free part M^{free} (with trivial action).

Suppose we have a short exact sequence of linearized modules $0 \rightarrow M'' \rightarrow M \xrightarrow{\phi} M' \rightarrow 0$. We wish to show that any invariant $m' \in M'$ is the image of an invariant element of M . Since ϕ is surjective, we have that $m' = \phi(m_f + m_t)$, where m_t is torsion and m_f is invariant. Since torsion sheaves have trivial cohomology, any invariant torsion element which is the image of a torsion element is actually the image of an *invariant* torsion element, so $\phi(m_t) = \phi(n_t)$ for some invariant torsion element $n_t \in M$. Then $m_f + n_t$ is invariant and $\phi(m_f + n_t) = m'$. This completes the proof that $BG \rightarrow \operatorname{Spec} k[[t]]$ is a good moduli space morphism.

Now take any vector bundle over the origin with non-trivial \mathbb{G}_m action. By Lemma 13, this extends to a unique vector bundle on \widehat{BG} , but we've seen that there is no torsion-free coherent sheaf on BG with non-trivial action on the special fiber. \diamond

Remark 25. A similar example gives a counterexample to [Alp10, Conjecture 1]. Let

$$G' = \text{Proj}(\mathbb{C}[t, x, y, z]/(zy^2 - x^2(x + z) - tz^3))$$

where t has degree 0 and x, y , and z have degree 1. Let G be the largest subscheme of G' over which the map to $\mathbb{A}^1 = \text{Spec } \mathbb{C}[t]$ is smooth. By [Sil94, IV Theorem 5.3(c)], G is a relative group over \mathbb{A}^1 . Let $\mathcal{X} = BG$. This \mathcal{X} is finitely presented over $\text{Spec } \mathbb{C}$, and the image of the origin in \mathbb{A}^1 is a closed point x with stabilizer \mathbb{G}_m . If [Alp10, Conjecture 1] were true, there would be an algebraic space Y with a point y and an étale representable morphism $f: [Y/\mathbb{G}_m] \rightarrow \mathcal{X}$ sending y to x , inducing an isomorphism of stabilizers. As f is étale, its image is open, so the image contains some closed point of \mathcal{X} whose stabilizer is an elliptic curve. By [GS11, Proposition 3.2], f induces finite-index inclusions of stabilizers. But no subgroup of \mathbb{G}_m can possibly be a finite-index subgroup of an elliptic curve.

It is possible that [Alp10, Conjecture 1] holds for stacks with affine stabilizers.

Remark 26. Taking $\mathcal{X} = BG$ and $\mathcal{X}' = B\mathbb{G}_m$ over $A = k[[t]]$, Example 24 shows that the natural map

$$\text{Hom}_A(\mathcal{X}, \mathcal{X}') \rightarrow \text{Hom}_A(\widehat{\mathcal{X}}, \widehat{\mathcal{X}}')$$

is not necessarily an equivalence of categories. The complex analytic analogue of this natural map is an equivalence of categories if \mathcal{X} is a proper Deligne-Mumford stack and \mathcal{X}' is either a quasi-compact algebraic stack with affine diagonal [Lur04, Theorem 1.1] or a locally of finite type Deligne-Mumford stack with quasi-compact and quasi-separated diagonal [Hal11, Theorem 1].

Remark 27. It is difficult to imagine an example of a stack \mathcal{X} with affine diagonal and good moduli space $\text{Spec } A$ which is not a quotient stack (i.e. does not satisfy (quot')) étale locally on $\text{Spec } A$. Likely candidates, such as non-trivial \mathbb{G}_m -gerbes, do not work (see Remark 32). If no such stack exists, then Theorems 1 and 20 show that formal GAGA holds provided that \mathcal{X} has affine diagonal.

Conjecture 28. *Suppose $\phi: \mathcal{X} \rightarrow \text{Spec } A$ is a good moduli space morphism, where A is a complete Noetherian local ring and ϕ is of finite type. If \mathcal{X} has affine diagonal, then the completion functor $\text{Coh}(\mathcal{X}) \rightarrow \text{Coh}(\widehat{\mathcal{X}})$ is an equivalence of categories.*

Formal GAGA may hold even if \mathcal{X} does not have affine diagonal, but it is usually uninteresting. For example, for any elliptic curve $E \rightarrow \text{Spec } A$, formal GAGA holds for $BE \rightarrow \text{Spec } A$ since all coherent sheaves on BE are pulled back from $\text{Spec } A$.

5. APPLICATION TO THE LOCAL QUOTIENT STRUCTURE OF GOOD MODULI SPACES

Recall that a stack \mathcal{X} is a *quotient stack* if it is the stack quotient of an algebraic space by a subgroup of GL_n for some n (i.e. if (quot') holds).

Proposition 29. *Let \mathcal{X} be a stack with good moduli space $\phi: \mathcal{X} \rightarrow X$, with ϕ of finite type and X locally Noetherian. Let $x \in X$ be a point such that the fiber \mathcal{X}_0 over x is a quotient stack. Suppose that formal GAGA holds for $\widetilde{\mathcal{X}} = \mathcal{X} \times_X \text{Spec } \widehat{\mathcal{O}}_{X,x} \rightarrow \text{Spec } \widehat{\mathcal{O}}_{X,x}$.⁷ Then there exists an étale neighborhood $X' \rightarrow X$ of x such that $\mathcal{X} \times_X X'$ is a quotient stack.*

⁷If $x \in X$ has an open neighborhood which is a scheme, $\mathcal{O}_{X,x}$ denotes the ordinary (Zariski) local ring. If not, then $\mathcal{O}_{X,x}$ denotes the local ring of some preimage under an étale cover by a scheme. Any two étale covers have a common refinement, so by Theorem 20, it doesn't matter which cover is used. (Also see Remark 30.)

$$\begin{array}{ccccccccc}
\widehat{\mathcal{X}} & \longrightarrow & \mathcal{X}^{loc} & \longrightarrow & \mathcal{X}^h & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
X^{loc} & = & X^{loc} & \longrightarrow & X^h & \longrightarrow & X' & \longrightarrow & X \\
& & \parallel & & \parallel & & \parallel & & \\
& & \mathrm{Spec} \widehat{R}^h & & \mathrm{Spec} R^h & & \mathrm{Spec} R_j & &
\end{array}$$

Proof. The question is étale local on X , and by Theorem 20 the hypothesis is étale local on X , so we may assume that $X = \mathrm{Spec} R$ is an affine scheme. Let $X^h = \mathrm{Spec} R^h$, where R^h is the strict Henselization of R at x , and let $X^{loc} = \mathrm{Spec} \widehat{R}^h$. Let \mathcal{X}^{loc} and \mathcal{X}^h denote the pullback of \mathcal{X} to X^{loc} and X^h , respectively. For a sheaf \mathcal{F} on \mathcal{X} (or \mathcal{X}^h), let \mathcal{F}^h and \mathcal{F}^{loc} denote the pullback of \mathcal{F} to \mathcal{X}^h and \mathcal{X}^{loc} , respectively. Let $\widehat{\mathcal{X}}$ be the completion of \mathcal{X}^{loc} with respect to the maximal ideal of \widehat{R}^h .

The closed substack $\mathcal{X}_0 \subseteq \widehat{\mathcal{X}}$ is a quotient stack, so its unique closed point has affine stabilizer, and it has the resolution property by Remark 15. By assumption, (GAGA) holds for $\widehat{\mathcal{X}} \rightarrow \mathrm{Spec} \widehat{\mathcal{O}}_{X,x}$, so by Theorem 1(ii), $\widehat{\mathcal{X}} = [U/GL_n]$ for an affine scheme U . Since $\mathrm{Spec} \widehat{R}^h \rightarrow \mathrm{Spec} \widehat{\mathcal{O}}_{X,x}$ is an affine morphism, $\mathcal{X}^{loc} \rightarrow \widehat{\mathcal{X}}$ is affine, so $U^{loc} = U \times_{\widehat{\mathcal{X}}} \mathcal{X}^{loc}$ is an affine scheme and $\mathcal{X}^{loc} = [U^{loc}/GL_n]$. By Remark 15, \mathcal{X}^{loc} has the resolution property.

Next we show that \mathcal{X}^h has the resolution property. Let \mathcal{F} be a coherent sheaf on \mathcal{X}^h . By the previous paragraph, there is a vector bundle \mathcal{V}^{loc} on \mathcal{X}^{loc} with a surjection to \mathcal{F}^{loc} . By [LMB00, Proposition 4.18(i)], the stack of rank n vector bundles on \mathcal{X} , $\mathcal{H}om(\mathcal{X}, BGL_n)$, is locally of finite presentation over X . By Artin approximation [Art69, Theorem 1.12], there exists a vector bundle \mathcal{V} on \mathcal{X}^h such that the pullback of \mathcal{V} to \mathcal{X}_0 is the same as the pullback of \mathcal{V}^{loc} to \mathcal{X}_0 . By Lemma 13 and Theorem 1(i), the pullback of \mathcal{V} to \mathcal{X}^{loc} is isomorphic to \mathcal{V}^{loc} . Since $\mathcal{H}om(\mathcal{V}, \mathcal{F})$ is locally of finite presentation and the substack of surjections is open in $\mathcal{H}om(\mathcal{V}, \mathcal{F})$ [Lie06, Lemma 2.2.2], the substack of surjections is locally of finite presentation. By Artin approximation, there exists a surjection $\mathcal{V} \rightarrow \mathcal{F}$. This proves that \mathcal{X}^h has the resolution property.

Let \mathcal{X}_0^h denote the closed fiber of \mathcal{X}^h . The morphism $\mathcal{X}_0^h \rightarrow \mathcal{X}_0$ is a representable morphism to a quotient stack. If $\mathcal{X}_0 = [U/G]$, then $\mathcal{X}_0^h = [(U \times_{\mathcal{X}_0} \mathcal{X}_0^h)/G]$, so \mathcal{X}_0^h is a quotient stack. In particular, the closed point of \mathcal{X}^h has affine stabilizer, so by Remark 15, $\mathcal{X}^h = [P^h/GL_n]$ for some affine scheme P^h . The GL_n -torsor $P^h \rightarrow \mathcal{X}^h$ corresponds to a representable map $p^h: \mathcal{X}^h \rightarrow BGL_n$. Since $\mathcal{H}om(\mathcal{X}, BGL_n)$ is locally of finite presentation over X^h and since $R^h = \varinjlim R_i$, where the limit runs over all étale neighborhoods $X_i = \mathrm{Spec} R_i \rightarrow \mathrm{Spec} R$ of x , we have that p^h is the pullback of some $p_i: \mathcal{X}_i = \mathcal{X} \times_X X_i \rightarrow BGL_n$. Let $\mathcal{Q}_i \rightarrow \mathcal{X}_i$ be the corresponding GL_n -torsor. To finish the proof, it suffices to show there exists an étale neighborhood $X_j \rightarrow X_i$ such that $\mathcal{Q}_j = \mathcal{Q}_i \times_{X_i} X_j$ is an affine scheme.

Since X is locally Noetherian, X^h is Noetherian [EGA IV, Proposition 18.8.8(iv)]. As P^h is finite type over X^h , it is finitely presented over X^h , so there exists an étale neighborhood $X_{j_0} \rightarrow X_i$ and an affine scheme P_{j_0} over X_{j_0} such that $P^h \cong P_{j_0} \times_{X_{j_0}} X^h$. Let $\mathcal{Q}_{j_0} = \mathcal{Q}_i \times_{X_i} X_{j_0}$. By [LMB00, Proposition 4.18(i)], $\mathcal{H}om_{X_{j_0}}(\mathcal{Q}_{j_0}, P_{j_0})$ and $\mathcal{H}om_{X_{j_0}}(P_{j_0}, \mathcal{Q}_{j_0})$ are locally of finite presentation over X_{j_0} , so there exists an étale neighborhood $X_{j_1} \rightarrow X_{j_0}$ such that the isomorphism $f: \mathcal{Q}_i \times_X X^h = \mathcal{Q}_{j_0} \times_{X_{j_0}} X^h \rightarrow P_{j_0} \times_{X_{j_0}} X^h = P^h$ and its inverse g are the pullbacks of maps f_1 and g_1 which are defined over X_{j_1} . By [LMB00, Proposition 4.18(i)], there is an étale neighborhood $X_j \rightarrow X_{j_1}$ such that the compositions $f_1 \circ g_1$ and $g_1 \circ f_1$ pull back to the identities over X_j . This shows that $\mathcal{Q}_j \cong P_j$ is an affine scheme, as desired. \square

Remark 30. In the proof of Proposition 29, the formal GAGA hypothesis is only used to show that \mathcal{X}^{loc} has the resolution property. If this can be obtained in some other way (e.g. if formal GAGA holds for $\mathcal{X}^{loc} \rightarrow \widehat{\mathrm{Spec} \mathcal{O}_{X,x}^h}$), the rest of this proof works as above.

Because of results like Proposition 29, and more generally because of the strategy presented in Remark 4, it is desirable to have a classification of stacks which have a point as a good moduli space. It is not known which such stacks are quotient stacks.

Question 31. Does there exist a good moduli space morphism $\mathcal{Y} \rightarrow \mathrm{Spec} k$, with k a separably closed field, such that \mathcal{Y} has affine diagonal but is not a quotient stack?

Remark 32. One natural source of examples is non-trivial gerbes. By [EHKV01, Example 3.12] there are \mathbb{G}_m -gerbes which are not quotient stacks. If \mathcal{X} is a \mathbb{G}_m -gerbe over a Noetherian scheme X , then \mathcal{X} is a quotient stack if and only if its class in $H^2(X, \mathbb{G}_m)$ is in the image of the Brauer map $\mathrm{Br}(X) \rightarrow H^2(X, \mathbb{G}_m)$ [EHKV01, Theorem 3.6]. However, if $X = \mathrm{Spec} A$, where A is a complete local ring (e.g. a field), then by [Mil80, Corollary IV.2.12],⁸ the natural map $\mathrm{Br}(X) \rightarrow H^2(X, \mathbb{G}_m)$ is an isomorphism, so any \mathbb{G}_m -gerbe over $\mathrm{Spec} A$ is a quotient stack.

Remark 33. Another candidate counterexample is $\mathfrak{M}_0^{\leq m}$, the moduli stack of genus 0 prestable (i.e. nodal) curves with at most m nodes. Over any field K and for any $m \geq 2$, $\mathfrak{M}_0^{\leq m}$ is not a quotient stack [Kre, Proposition 5.2]. The map $\mathfrak{M}_0^{\leq m} \rightarrow \mathrm{Spec} K$ is universal for maps to algebraic spaces. However, it is not a good moduli space map since closed points of $\mathfrak{M}_0^{\leq m}$ can have non-reductive stabilizers: each outer leaf of any tree T of smooth rational curves contributes a copy of $\mathrm{Aut} \mathbb{A}^1 \cong \mathbb{G}_m \ltimes \mathbb{G}_a$ to $\mathrm{Aut} T$. But by [Alp09, Proposition 12.14], the stabilizers at closed points of a stack which has a good moduli space are linearly reductive.

A promising variant is $\mathfrak{M}_{0,n}^{\leq m}$, the moduli stack of marked genus 0 prestable curves with n marked points and at most m nodes, such that each component has at least two marks/nodes. The closed points of this stack have linearly reductive stabilizers, and the stack is non-empty for $n \geq 2$. For $m \geq 2$, we believe a modification of Kresch's argument shows that $\mathfrak{M}_{0,n}^{\leq m}$ is not a quotient stack.⁹ For $n \geq 3$ and $m \geq 2$, there are curves which isotrivially degenerate to multiple closed points, so $\mathfrak{M}_{0,n}^{\leq m}$ cannot have a good moduli space by [Alp09, Proposition 4.16(iii)]. The stack $\mathfrak{M}_{0,2}^{\leq 2}$ has a unique closed point (topologically, it is a chain of 3 points) and the map to a point is universal for maps to algebraic spaces. If this map is a good moduli space morphism, it would answer Question 31 affirmatively.

Remark 34. Suppose $\mathcal{X} \rightarrow \mathrm{Spec} A$ is a good moduli space as in §1, with $k = A/\mathfrak{m}$ separably closed. Suppose \mathcal{X} has affine diagonal, and satisfies [Alp10, Conjecture 1] (by Remark 25, we cannot expect this unless \mathcal{X} has affine stabilizers). Let G_x be the stabilizer of the unique closed point x of \mathcal{X} . Then there is a representable étale morphism $f: \mathcal{W} = [U/G_x] \rightarrow \mathcal{X}$ and a point $w \in \mathcal{W}(k)$ such that the induced map $\mathrm{Aut}_{\mathcal{W}(k)}(w) \rightarrow \mathrm{Aut}_{\mathcal{X}(k)}(x) = G_x$ is an isomorphism. Suppose the strong form of this conjecture holds (i.e. that we may take $U = \mathrm{Spec} R$ to be affine; see [Alp10, second paragraph after Conjecture 1]).

⁸Note that in contrast to [EHKV01], Milne defines $\mathrm{Br}'(X) = H^2(X, \mathbb{G}_m)$ (p. 147).

⁹Here is a sketch of the argument. There is an open immersion $\mathfrak{M}_{0,n}^{\leq 2} \subseteq \mathfrak{M}_{0,n}^{\leq m}$, so it suffices to show $\mathfrak{M}_{0,n}^{\leq 2}$ is not a quotient stack. There is a representable morphism $\widetilde{\mathfrak{M}}_{0,n}^{\leq 2} \rightarrow \mathfrak{M}_{0,n}^{\leq 2}$ from the stack in which the points are *labeled*, so it suffices to check that the former is not a quotient stack. There is a morphism $\widetilde{\mathfrak{M}}_{0,2}^{\leq 2} \rightarrow \mathfrak{M}_{0,2}^{\leq 2}$ given by adding points in a prescribed fashion, which is the trivial \mathbb{G}_m -gerbe over its image (for $n \geq 3$), so it suffices to check $\widetilde{\mathfrak{M}}_{0,2}^{\leq 2}$ is not a quotient stack. A straightforward modification of the proof of [Kre, Proposition 5.2] shows that $\widetilde{\mathfrak{M}}_{0,2}^{\leq 2}$ is not a quotient stack.

(This argument was shown to us by Jarod Alper.) Let $\mathcal{W} = [\mathrm{Spec} R/G_x] \rightarrow \mathcal{X}$ be as above. By [Alp09, Theorem 5.1], the induced map on good moduli spaces $\mathrm{Spec} R^{G_x} \rightarrow \mathrm{Spec} A$ is étale. Since A is complete with separably closed residue field, the component of $\mathrm{Spec} R^{G_x}$ containing the image of w must be isomorphic to $\mathrm{Spec} A$, so after shrinking $\mathrm{Spec} R$, we may assume $f: \mathcal{W} \rightarrow \mathcal{X}$ induces an isomorphism of good moduli spaces.

We claim that f is an isomorphism. Since f is étale, its image is open. Any open set containing the unique closed point x of \mathcal{X} is all of \mathcal{X} , so f is an étale cover. We may check that a morphism is an isomorphism étale locally on the base, so it suffices to show that the projection $p_1: \mathcal{W} \times_{\mathcal{X}} \mathcal{W} \rightarrow \mathcal{W}$ is an isomorphism. By [Alp09, Proposition 4.7(i)] $\mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ has good moduli space $\mathrm{Spec} A \times_{\mathrm{Spec} A} \mathrm{Spec} A = \mathrm{Spec} A$, so it has a unique closed point. The diagonal $\mathcal{W} \rightarrow \mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ has this closed point in its image. As the diagonal is a section of an étale morphism, it is an open immersion, so it is an isomorphism.

The strong form of [Alp10, Conjecture 1] for stacks with affine diagonal therefore answers Question 31 negatively: if $A = k$ is a separably closed field, the above argument shows that \mathcal{X} is a quotient stack.

Remark 35. If the strong form of [Alp10, Conjecture 1] for stacks with affine diagonal is true, the following argument shows that Conjecture 28 is true. In this case, the formal GAGA hypothesis in Proposition 29 may be replaced by the hypothesis that \mathcal{X} has affine diagonal.

Let G_x denote the stabilizer of the closed point x of \mathcal{X} . By [Alp09, Proposition 12.14], G_x is linearly reductive. Let $\mathcal{W} = [\mathrm{Spec} R/G_x] \rightarrow \mathcal{X}$ and $w \in \mathcal{W}$ be as in [Alp10, Conjecture 1]. By [Alp09, Theorem 5.1], the map on good moduli spaces $\mathrm{Spec} R^{G_x} \rightarrow \mathrm{Spec} A$ is étale, so after shrinking $\mathrm{Spec} R$, we may assume $\mathrm{Spec} R^{G_x} \rightarrow \mathrm{Spec} A$ is a finite étale extension. As \mathcal{W} and $\mathcal{X} \times_{\mathrm{Spec} A} \mathrm{Spec} R^{G_x}$ are both étale over \mathcal{X} , the induced morphism $\mathcal{W} \rightarrow \mathcal{X} \times_{\mathrm{Spec} A} \mathrm{Spec} R^{G_x}$ is étale. This morphism induces an isomorphism on good moduli spaces, and the image contains the unique closed point of $\mathcal{X} \times_{\mathrm{Spec} A} \mathrm{Spec} R^{G_x}$. By the argument in Remark 34, the map is an isomorphism. As \mathcal{W} is a quotient stack, formal GAGA holds for $\mathcal{W} \rightarrow \mathrm{Spec} R^{G_x}$ by Theorem 1(ii). By Theorem 20, formal GAGA holds for $\mathcal{X} \rightarrow \mathrm{Spec} A$.

REFERENCES

- [Alp09] Jarod Alper. Good moduli spaces for Artin stacks. Oct 2009. <http://arxiv.org/abs/0804.2242v3>.
- [Alp10] Jarod Alper. On the local quotient structure of Artin stacks. *J. Pure Appl. Algebra*, 214(9):1576–1591, 2010.
- [AOV08] Dan Abramovich, Martin Olsson, and Angelo Vistoli. Tame stacks in positive characteristic. *Ann. Inst. Fourier (Grenoble)*, 58(4):1057–1091, 2008.
- [Art69] M. Artin. Algebraic approximation of structures over complete local rings. *Inst. Hautes Études Sci. Publ. Math.*, (36):23–58, 1969.
- [ASvdW10] Jarod Alper, David Ishii Smyth, and Frederick van der Wyck. Weakly proper moduli stacks of curves, 2010. <http://arxiv.org/abs/1012.0538v2>.
- [Con] Brian Conrad. Formal GAGA for Artin stacks. <http://math.stanford.edu/~conrad/papers/formalgaga.pdf>.
- [EHKV01] Dan Edidin, Brendan Hassett, Andrew Kresch, and Angelo Vistoli. Brauer groups and quotient stacks. *Amer. J. Math.*, 123(4):761–777, 2001.
- [EGA IV] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967.
- [Eis95] David Eisenbud. *Commutative algebra, with a view toward algebraic geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
- [FC90] Gerd Faltings and Ching-Li Chai. *Degeneration of abelian varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990. With an appendix by David Mumford.
- [FGI⁺05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry, Grothendieck’s FGA explained*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

- [GS11] Anton Geraschenko and Matthew Satriano. Toric stacks II: Intrinsic characterization of toric stacks. 2011. <http://arxiv.org/abs/1107.1907>.
- [GIT] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.
- [Hal11] Jack Hall. Generalizing the GAGA principle, 2011. <http://arxiv.org/abs/1101.5123v2>.
- [HR10] Jack Hall and David Rydh. The Hilbert stack, 2010. <http://arxiv.org/abs/1011.5484v1>.
- [Knu71] Donald Knutson. *Algebraic spaces*. Lecture Notes in Mathematics, Vol. 203. Springer-Verlag, Berlin, 1971.
- [Kre] A. Kresch. Flattening stratification and the stack of partial stabilisations of prestable curves. *Bull. London Math. Soc.*, to appear.
- [Lie06] Max Lieblich. Remarks on the stack of coherent algebras. *Int. Math. Res. Not.*, pages Art. ID 75273, 12, 2006.
- [LO09] Max Lieblich and Brian Osserman. Functorial reconstruction theorems for stacks. *J. Algebra*, 322(10):3499–3541, 2009.
- [LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.
- [Lur04] Jacob Lurie. Tannaka duality for geometric stacks, 2004. <http://arxiv.org/abs/math/0412266v2>.
- [Mil80] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.
- [Ols05] Martin C. Olsson. On proper coverings of Artin stacks. *Adv. Math.*, 198(1):93–106, 2005.
- [SGA3] *Schémas en groupes. Tome 1: Propriétés générales des schémas en groupes*. Lecture Notes in Mathematics, Vol. 151. Springer-Verlag, Berlin, 1970. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 151.
- [SGA4] *Théorie des topos et cohomologie étale des schémas..* Lecture Notes in Mathematics, Vols. 269, 270, and 305. Springer-Verlag, Berlin, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat.
- [Sil94] Joseph H. Silverman. *Advanced topics in the arithmetic of elliptic curves*, volume 151 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.
- [Stacks] The Stacks Project Authors. *Stacks Project*. <http://stacks.math.columbia.edu/>
- [Tot04] Burt Totaro. The resolution property for schemes and stacks. *J. Reine Angew. Math.*, 577:1–22, 2004.